Math 132: Differential Topology

§ Sard's theorem

Thm (Sard's theorem)

Let $f: M \to N$ be a smooth map, and let

$$C = \{ x \in M \mid \text{rank } df_x < \dim N \}$$

be the set of critical points.

Then, the set of critical values, $f(C)$, has measure zero in N.

proof)

Thanks to second-countability of manifolds, it suffices to prove it for $f: U \to \mathbb{R}^p$, $U \overset{\text{open}}{\subset} \mathbb{R}^n$.

We proceed by induction on n. Note, it is trivial for $n=0$.

Let $C_1 = \{ x \in U \mid df_x = 0 \}$ and more generally

$$C_k = \{ x \in U \mid \frac{\partial^i f}{\partial x_{s_1} \cdots \partial x_{s_i}}(x) = 0 \text{ for all } \overset{i \le k \text{ and}}{1 \le s_1, \cdots, s_i \le n} \}.$$

Thus we have a descending sequence of closed sets

$$C \supset C_1 \supset C_2 \supset \cdots .$$

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We will prove the theorem in three steps:

Step 1: $f(C-C_1)$ has measure zero.

Step 2: $f(C_k - C_{k+1})$ has measure zero, for $k \geq 1$.

Step 3: $f(C_k)$ has measure zero for k sufficiently large.

proof of Step 1)

We may assume $p \geq 2$, since $C = C_1$ when $p=1$.

We will use

Thm (Fubini)

A measurable set $A \subset \mathbb{R}^p = \mathbb{R} \times \mathbb{R}^{p-1}$ must have measure zero if it intersects each hyperplane (const.) $\times \mathbb{R}^{p-1}$ in a set of $(p-1)$-dimensional measure zero.

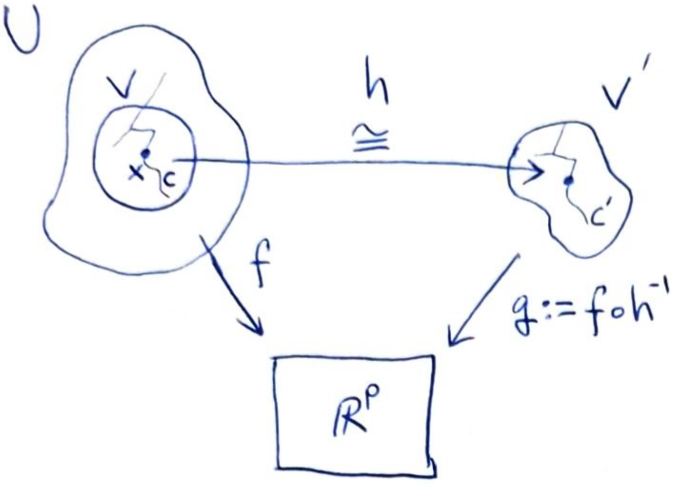
For each $x \in C - C_1$, we will find an open neighborhood $V \subset \mathbb{R}^n$ so that $f(V \cap C)$ has measure zero.

Since $x \notin C_1$, some partial derivative, say $\frac{\partial f_1}{\partial x_1}$, is nonzero at x.

Consider the map $h: U \to \mathbb{R}^n$

$$x \mapsto (f_1(x), x_2, \cdots, x_n).$$

Since dh_x is nonsingular, it is a local diffeomorphism; it maps some neighborhood V of x diffeomorphically onto an open set $V' \subset \mathbb{R}^n$.



Set $g := f \circ h^{-1}$. Then the set C' of critical points of g is $h(V \cap C)$, and the set $g(C')$ of critical values of g is $f(V \cap C)$.

Note, g carries hyperplanes into hyperplanes; let

$$g^t : (t \times \mathbb{R}^{n-1}) \cap V' \longrightarrow t \times \mathbb{R}^{p-1}$$

denote the restriction of g. Moreover, since

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & \left(\frac{\partial g_i^t}{\partial x_j}\right) \end{array}\right),$$

a point of $t \times \mathbb{R}^{n-1}$ is critical for g^t iff it is critical for g.

By induction hypothesis, the set of critical values of g^t has measure zero in $t \times \mathbb{R}^{p-1}$. Hence, by Fubini's theorem, the set

$$g(C') = f(V \cap C)$$

has measure zero, which completes Step 1.

proof of Step 2) (Similar to Step 1)

For each $x \in C_k - C_{k+1}$, some $(k+1)$-st derivative is nonzero,

so let's say $w(x) = \dfrac{\partial^k f_r}{\partial x_{s_2} \cdots \partial x_{s_{k+1}}}$ vanishes at x but $\dfrac{\partial w}{\partial x_1}$ does not.

Then the map $h : U \to \mathbb{R}^n$, $x \mapsto (w(x), x_2, \cdots, x_n)$ carries some neighborhood V

of x diffeomorphically onto an open set $V' \subset \mathbb{R}^n$.

Note, h carries $C_k \cap V$ into the hyperplane $0 \times \mathbb{R}^{n-1}$.

Hence, we can consider $g := f \circ h^{-1} : V' \to \mathbb{R}^p$

and its restriction $\bar{g} : (0 \times \mathbb{R}^{n-1}) \cap V' \to \mathbb{R}^p$.

By induction, the set of critical values of $\bar{g}$ has measure 0 in $\mathbb{R}^p$,

and in particular, $\bar{g} \circ h(C_k \cap V) = f(C_k \cap V)$ has measure zero.

proof of Step 3)

Let $I^n \subset U$ be a cube with edge δ. We will prove that if k is sufficiently large,

$f(C_k \cap I^n)$ has measure zero. (constant dependent only on f and I^n)

By Taylor's theorem, $f(x+h) = f(x) + R(x,h)$ where $\|R(x,h)\| \leq c \, \|h\|^{k+1}$

for $x \in C_k \cap I^n$, $x+h \in I^n$.

Now, subdivide I^n into r^n cubes of edge $\frac{\delta}{r}$.

Then, for each cube I' in the subdivision containing a point $x \in C_k$, any other point of I' can be written as $x+h$ with $\|h\| \leq \sqrt{n}\frac{\delta}{r}$,

so $$\text{Volume}(f(C_k \cap I')) \leq \left(2c\left(\sqrt{n}\frac{\delta}{r}\right)^{k+1}\right)^p.$$

Therefore,

$$\text{Volume}(f(C_k \cap I^n)) \leq r^n \cdot \left(2c\left(\sqrt{n}\frac{\delta}{r}\right)^{k+1}\right)^p$$

$$= c' \cdot r^{n-(k+1)p},$$

holds for any $r \in \mathbb{Z}_{>0}$, so we can send $r \to \infty$.

and hence if $k+1 > \frac{n}{p}$, $f(C_k \cap I^n)$ must have measure zero.

This completes the proof of Sard's theorem. ■